Consequences of a Killing symmetry in spacetime's local structure

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Abstract. In this paper we discuss the consequences of a Killing symmetry on the local geometrical structure of four-dimensional spacetimes. We have adopted the point of view introduced in recent works where the exterior derivative of the Killing plays a fundamental role. Then, we study some issues related with this approach and clarify why in many circumstances its use has advantages with respect to other approaches. We also extend the formalism developed in the case of vacuum spacetimes to the general case of an arbitrary energy-momentum content. Finally, we illustrate our framework with the case of spacetimes with a gravitating electromagnetic field.

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1. Introduction

The imposition of symmetries is and has been one the most successful ways of simplifying the theory in order to deal with the essence of physical problems. In the case of General Relativity they have been extensively used to simplify Einstein's field equations, and in this way to obtain exact solutions describing the gravitational field of the idealized situation. Of course, due to the non-linear character of the field equations, assuming many symmetries can lead to solutions which may not capture the actual behaviour of the physical system they are meant to represent. Then, it is possible that *deviations* from the idealized situation behave in a very different way. Nevertheless, we know of many systems whose dynamics is driven by the gravitational interaction and where there is a symmetry that is present in the spacetime region of interest. Then, it is important to know how the imposition of a symmetry affects to the structure of the spacetime and which can be the consequences for a specific problem.

In this paper we will consider spacetimes with only one Killing symmetry, which can describe many physical situations of interest. Examples of present interest are provided by physical systems in quasi-equilibrium configurations, like neutron-star

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binaries around the innermost stable circular orbit (ISCO), which are of great interest in the study of gravitational wave emission by binary systems. In this context, methods to construct such configurations in order to locate the ISCO rely on the assumption of the existence of an approximate Killing vector [1]. Other problems of interest in which a symmetry is present are those involving axisymmetric configurations, stationary systems, etc.

There are many papers in the literature dealing with spacetimes possesing Killing vector fields (KVFs hereafter). Most of these papers are devoted to the search of exact solutions of Einstein's field equations, and KVFs were considered as a way of simplifying the problem (see [2]). Then, the main aim was to develop and apply techniques to solve the Einstein equations when a group of Killing symmetries was imposed. Despite the huge amount of work done there are few studies that concentrate only on the study of spacetimes containing just one KVF. Then, few is known about the geometrical properties of a Killing symmetry in connection with the spacetime geometry. Examples of works where formalisms for one KVF are introduced are the works by Collinson and French [3] (for conformal KVFs) and by Perjés [4]. In the first work the equations (including the Killing equations) are written using the Newman-Penrose formalism [5], which leads to simplifications when the NP basis is adapted to the algebraic structure of the spacetime. However, it has the limitation that it does not introduce any quantity adapted to the (conformal) KVF. Then, by one hand it can be used for groups of (conformal) KVFs but, on the other hand, it cannot deal with the particular characteristics of the symmetry. In contrast, the work by Perjés combines in a powerful way the spinor formalism (the NP formalism is a byproduct of it) and quantities describing the structure of the symmetry. However, since it uses quantities related to the Ernst potential, the generalization to general spacetimes with one KVF is not obvious at all.

Recently, we introduced [6, 7] a new approach to the study of vacuum spacetimes with one KVF which is adapted to the structure of the Killing symmetry and which can be easily generalized to any matter content (this is one of the subjects of the present paper). As we showed in the previous works [6, 7], this approach naturally leads to a classification [7] of the spacetimes with at least one symmetry, which is based on the properties of the KVF and on the relations of its algebraic structure with that of the spacetime (the algebraic structure of the Weyl tensor). Moreover, in [6, 7] we showed the potential of this approach by proving a number of new results in the vacuum case (see [14] for an error in one of them). Generalizations of this formalism have recently appeared for the case of homotheties [15] and conformal KVFs [16].

The approach introduced in [6, 7] considers the exterior derivative of the Killing, which we will call the *Papapetrou field* after its introduction in [8] (sometimes also called the Killing 2-form or the Killing bivector [9, 10]), and its algebraic structure, as main objects of the study. The Papapetrou field has been used before in the search of exact solutions (see, e.g., [11]) and in the study of external magnetic fields in black holes [12]. As we discussed in detail in [6, 7], one main advantage of this

approach is that it provides a framework in which one can establish, in a natural way, connections between the particular characteristics of the symmetry (the structure of the Papapetrou field) and the algebraic structure of the spacetime (the Petrov type of the Weyl tensor [13]) which, in particular, leads to a detailed classifications of the spacetimes with an isometry [7]. Moreover, with the introduction of the Papapetrou field as a main quantity we constructed a new formalism for the study of vacuum spacetimes with an KVF [7]. It is an extension of the Newman-Penrose formalism [5], where new variables associated with the KVF and its Papapetrou field are introduced. Within this formalisms, the integrability conditions for the components of the KVF are analyzed from a new perspective. Using this new point of view it was shown in [7] that from the integrability conditions one gets expressions for all the Weyl tensor components in terms of the other variables, reducing in this way the number of equations to be considered (we avoid the use of the second Bianchi identities).

In this paper we analyze in detail the structure of this approach and the reasons why it has advantages over other approaches, and we extend the formalism to spacetimes with a KVF and with a generic energy-momentum content. The plan of the paper is the following: In section 2 we present the main ideas and justify why the adoption of the Papapetrou field as a main quantity means to have advantages when dealing with spacetimes with a KVF. At the same time we study in detail, from a new perspective, the integrability conditions for the components of the KVF. This part will also involve an explanation of previous results [6, 7] in a broader context. In section 3 we study the integrability conditions for the Papapetrou field, which satisfies Maxwell equations. In section 4 we extend the framework introduced in [7] for vacuum to general spacetimes. For the sake of clarity we will give the equations that come out from this formalism in the Appendix A, where they have been expressed using the Newman-Penrose formalism [5]. In section 5 we illustrate these developments with the case of spacetimes with a gravitating electromagnetic field, and in particular, with the case of the Kerr-Newman metric. We finish with some remarks and conclusions in section 6. Throughout this paper we adopt units in which $c = 8\pi G = 1$, and we will use the conventions and definitions introduced in [2] unless stated otherwise.

2. On the consequences of the integrability conditions for the Killing equations

From now on we will assume the spacetime is endowed with a non-null KVF $\boldsymbol{\xi}$

$$N = \mathrm{g}_{ab}\xi^a\xi^b \neq 0\,. \tag{1}$$

From the definition of a Killing symmetry it follows that $\boldsymbol{\xi}$ satisfies the Killing equations

$$\pounds_{\boldsymbol{\xi}}\mathrm{g}_{ab} = 0 \quad \Longleftrightarrow \quad \xi_{a;b} + \xi_{b;a} = 0\,, \tag{2}$$

where the semicolon denotes covariant differentiation. We can know about the Killing symmetries that a given spacetime admits by solving these equations. Of course, there are spacetimes that do not admit any Killing symmetry, in which case one finds that

the Killing equations have no solutions. We can investigate whether or not the Killing equations admit solutions in a given spacetime by studying their integrability conditions, which can be expressed in the following compact form

$$\pounds_{\boldsymbol{\xi}}\Gamma^a{}_{bc} = \pounds_{\boldsymbol{\xi}} R^a{}_{bcd} = \pounds_{\boldsymbol{\xi}} R^a{}_{bcd;e_1} = \pounds_{\boldsymbol{\xi}} R^a{}_{bcd;e_1e_2} = \cdots = 0\,, \tag{3}$$

where $\pounds_{\boldsymbol{\xi}}$ denotes Lie differentiation along $\boldsymbol{\xi}$, $\Gamma^a{}_{bc}$ are the Christoffel symbols, and $R^a{}_{bcd}$ the components of the Riemann curvature tensor. As we can see, the integrability conditions involve, in general, derivatives of the curvature tensor.

In recent works [6, 7] we introduced an alternative way of dealing with vacuum spacetimes admitting a Killing symmetry. The main object in this approach is the exterior derivative of the KVF

$$\boldsymbol{F} = \boldsymbol{d\xi} \quad \Rightarrow \quad F_{ab} = \xi_{b;a} - \xi_{a;b} = 2\xi_{b;a}\,,$$

where we have used the Killing equations (2). This quantity is important for several reasons. As it was firstly recognized by Papapetrou [8], a KVF $\boldsymbol{\xi}$ can always be seen as the vector potential generating an electromagnetic field, F_{ab}, satisfying Maxwell equations

$$F_{[ab;c]} = 0, \quad F^{ab}{}_{;b} = J^a\,, \tag{4}$$

where J^a is the conserved current given by

$$J^a = 2R^a{}_b\,\xi^b \quad \Rightarrow \quad J^a{}_{;a} = 0\,, \tag{5}$$

being $R_{ab} = R^c{}_{acb}$ the Ricci tensor. Apart from the Maxwell equations, F_{ab} satisfies the following relation

$$\pounds_{\boldsymbol{\xi}} F_{ab} = 0\,, \tag{6}$$

which is a direct consequence of the Killing equations and it is equivalent to

$$\xi^c F_{ab;c} = 0\,. \tag{7}$$

In the approach proposed in [6, 7] the components of the Papapetrou field F_{ab} are promoted to the level of variables of the formalism, like the components of the KVF. Then, taking into account the Killing equations (2), the equations for $\boldsymbol{\xi}$ are now

$$\xi_{b;a} = \tfrac{1}{2}F_{ab} \qquad \text{with} \qquad F_{ab} = F_{[ab]}\,. \tag{8}$$

In the Appendix A we have expressed them using the NP formalism.

Their integrability conditions are equivalent to the Ricci identities for the KVF $\boldsymbol{\xi}$

$$\xi_{a;dc} - \xi_{a;cd} = R_{abcd}\xi^b\,. \tag{9}$$

In the four-dimensional spacetimes of general relativity, and taking into account only the antisymmetry of the Riemann tensor in the first and second pair of indices ($R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$), expression (9) contains 24 independent equations. They can be split into four differentiated groups: (i) First Bianchi identities: $R_{[abc]d}\xi^d = 0 \Rightarrow \xi_{[ab;c]} = 0$. They contain four independent equations which lead to the first group of Maxwell equations (4). (ii) The trace: $\xi^a{}_{b;c} - \xi^a{}_{a;b} = R_{bd}\xi^d$. Four independent

components equivalent to the second set of Maxwell equations (4). (iii) $R_{abcd}\xi^a\xi^b = 0 \Rightarrow \xi^a(\xi_{a;dc} - \xi_{a;cd}) = 0$. This expression contains 6 independent equations, which in combination with those in the group (i), lead to the property (6,7) of F_{ab}. (iv) The rest, 10 independent equations. Introducing (8) for the derivatives of $\boldsymbol{\xi}$ and the decomposition of the Riemann tensor into its irreducible parts under the Lorentz group: the Weyl tensor C_{abcd}, the curvature tensor part not determined locally by the energy-momentum distribution, the traceless Ricci tensor $S_{ab} = R_{ab} - \frac{1}{4}\mathrm{g}_{ab}R$, and the scalar curvature $R = \mathrm{g}^{ab}R_{ab}$,

$$R_{abcd} = C_{abcd} \; + \; \mathrm{g}_{a[c}S_{d]b} - \mathrm{g}_{b[c}S_{d]a} \; + \; \tfrac{1}{12}(\mathrm{g}_{ac}\mathrm{g}_{bd} - \mathrm{g}_{ad}\mathrm{g}_{bc})R\,,$$

the remaining 10 equations in (9) can be written in the following form

$$C_{abcd}\xi^d = -\tfrac{1}{2}F_{ab;c} + \xi_{[a}R_{b]c} + \xi^d R_{d[a}\mathrm{g}_{b]c} - \tfrac{1}{3}\xi_{[a}\mathrm{g}_{b]c}R\,. \tag{10}$$

The right-hand side is made out of a term constructed only from the Papapetrou field and terms constructed from the Ricci tensor. The last ones can be written in terms of the matter variables, described by the energy-momentum tensor T_{ab}, through the Einstein field equations

$$G_{ab} = R_{ab} - \tfrac{1}{2}\mathrm{g}_{ab}R = T_{ab}\,, \tag{11}$$

where G_{ab} denotes the Einstein tensor. Then, we can think of equation (10) as a relation that will provide expressions for some components of the Weyl tensor in terms of F_{ab} and T_{ab}. As we have said before, equation (10) contains 10 independent equations, the same number of independent components of the Weyl tensor. Hence, we can determine all the components of this tensor from (10). However, this is true only in four-dimensional spacetimes. For D-dimensional spacetimes we would get an equation analogous to (10), with the same left-hand side and with the same terms on the right-hand side but with different numerical coefficients (see, e.g., [17] to obtain these coefficients). The number of independent components of the Weyl tensor in D dimensions is (see, e.g., [18])

$$C(D) = \tfrac{1}{12}D(D+1)(D+2)(D-3) \qquad (\text{for } D \geq 3)\,.$$

On the other hand, to find the number of independent components in the D-dimensional version of equation (10) we can write it as $C_{abcd}\xi^d = A_{abc}$. Then, taking into account the information coming from (i)-(iii), the tensor A_{abc} satisfy the following properties

$$A_{[abc]} = 0\,, \qquad A_{[ab]c} = A_{abc}\,, \qquad A_{abc}\xi^c = 0\,, \qquad A_{ab}{}^{a} = 0\,. \tag{12}$$

From these relations and the fact that we are considering KVFs with a non-vanishing norm N [see Eq. (1)], the number of independent components contained in Eq. (10) are

$$I(D) = \tfrac{1}{6}D(D+1)(2D-5)\,.$$

Hence, the number of independent components of the Weyl tensor that *cannot* be determined from the integrability conditions (10) is given by

$$\Delta(D) = C(D) - I(D) = \tfrac{1}{12}D(D^2-1)(D-4)\,.$$

Therefore, in the case of General Relativity, $D = 4$, we can determine completely the Weyl tensor $[\Delta(4) = 0]$ in terms of the Papapetrou field F_{ab} and the energy-momentum content [see Eq. (10)]. In higher dimensions‖ this is not true, and we can only determine a subset of the independent components of the Weyl tensor.

As a conclusion we can say that from the integrability conditions (10) and in $D = 4$ we can find an expression for the Weyl tensor in terms of the Papapetrou field and the energy-momentum content. We can find the explicit expression of the Weyl tensor by different means. Here we will use a procedure based on the fact that for a given non-null vector field we can associate with it what are called the electric and magnetic parts of the Weyl tensor. We are interested in the electric and magnetic parts associated with the KVF $\boldsymbol{\xi}$:

$$\begin{aligned} E_{ab}[\boldsymbol{\xi}] &= C_{acbd}\xi^c\xi^d \\ H_{ab}[\boldsymbol{\xi}] &= *C_{acbd}\xi^c\xi^d \qquad (*C_{abcd} = \tfrac{1}{2}\eta_{ab}{}^{ef}C_{cdef})\,, \end{aligned} \tag{13}$$

where $*$ denotes the dual operation and η_{abcd} the components of the completely antisymmetric volume four form. The two tensors (13) are symmetric, trace-free and orthogonal to the vector field they are associated with, the KVF in our case,

$$E_{(ab)}[\boldsymbol{\xi}] = E_{ab}[\boldsymbol{\xi}]\,,\ H_{(ab)}[\boldsymbol{\xi}] = H_{ab}[\boldsymbol{\xi}]\,, \quad E^a{}_a[\boldsymbol{\xi}] = H^a{}_a[\boldsymbol{\xi}] = 0\,, \quad E_{ab}[\boldsymbol{\xi}]\xi^b = H_{ab}[\boldsymbol{\xi}]\xi^b = 0\,.$$

Moreover, they contain all the information about the Weyl tensor. Indeed, each part has five independent components and the Weyl tensor has ten. We can reconstruct the Weyl tensor from the electric and magnetic parts and the KVF $\boldsymbol{\xi}$ through the following expression

$$C_{abcd} = \frac{1}{N^2}\xi^e\xi^g\left\{(\mathrm{g}_{abef}\mathrm{g}_{cdgh} - \eta_{abef}\eta_{cdgh})E^{fh}[\boldsymbol{\xi}] - (\mathrm{g}_{abef}\eta_{cdgh} + \eta_{abef}\mathrm{g}_{cdgh})H^{fh}[\boldsymbol{\xi}]\right\}\,, \tag{14}$$

where $\mathrm{g}_{abcd} = \mathrm{g}_{ac}\mathrm{g}_{bd} - \mathrm{g}_{ad}\mathrm{g}_{bc}$.

From the integrability conditions (10) we can compute the electric and magnetic parts associated with the KVF $\boldsymbol{\xi}$, the result is

$$E_{ab}[\boldsymbol{\xi}] = -\tfrac{1}{2}\xi^e\mathrm{g}_{ea}{}^{cd}A_{cdb}\,, \tag{15}$$

$$H_{ab}[\boldsymbol{\xi}] = -\tfrac{1}{2}\xi^e\eta_{ea}{}^{cd}A_{cdb}\,, \tag{16}$$

where the tensor A_{abc} was introduced above and has the properties shown in (12). In terms of F_{ab} and T_{ab} it has the following expression

$$A_{abc} = -\tfrac{1}{2}F_{ab;c} + \xi_{[a}T_{b]c} + \xi^dT_{d[a}\mathrm{g}_{b]c} - \tfrac{2}{3}\xi_{[a}\mathrm{g}_{b]c}T\,, \tag{17}$$

where $T = \mathrm{g}^{ab}T_{ab}$. Then, introducing this expression in equations (15,16) and finally into the Weyl tensor [Eq. (14)] we get

$$C_{ab}{}^{cd} = \frac{2}{N}\left\{A_{ab}{}^{[c}\xi^{d]} + A^{cd}{}_{[a}\xi_{b]} + 2\xi^eA_e{}^{[c}{}_{[a}\delta^{d]}{}_{b]}\right\}\,. \tag{18}$$

And this is how the integrability conditions for the components of the KVF supply an expression for the Weyl tensor in terms of the KVF, the Papapetrou field and the energy-momentum tensor.

‖ For $D \leq 3$ the Weyl tensor is identically zero.

We can insert (18) into the contracted second Bianchi identities (which in $D = 4$ contain the same information as the non-contracted ones)

$$C_{abcd}{}^{;d} = R_{c[a;b]} - \tfrac{1}{6} g_{c[a} R_{,b]} \,.$$

The result, using Einstein's equations (11), is

$$F_{ab;c}{}^{c} + C_{abcd} F^{cd} + 2J_{[a;b]} + \tfrac{1}{3} T F_{ab} = 0 \,. \tag{19}$$

One can reach this result starting from Maxwell equations (4), taking the covariant derivative of $F_{[ab;c]} = 0$ and using the Ricci identities. This means that (19) is an identity as long as F_{ab} satisfies Maxwell equations.

3. On the Integrability conditions for the Papapetrou field and other issues

In the previous section we have studied the integrability conditions for the components of the KVF. The next step is to study the equations for the Papapetrou field F_{ab}, i.e., Maxwell equations (4). Their integrability conditions have been studied in great detail in [19]. Here, we will apply their results to the case of a Papapetrou field. To that end, it is very convenient to consider the following two points. First, instead of using F_{ab}, we will work with the self-dual Papapetrou field

$$\tilde{F}_{ab} = F_{ab} + i * F_{ab} \qquad (*F_{ab} = \tfrac{1}{2}\eta_{abcd}F^{cd}) \,.$$

This will lead to more compact expression in our study. In particular, the Maxwell equations look simply as

$$\tilde{F}^{ab}{}_{;b} = J^a \,, \tag{20}$$

where J^a are the conserved sources, given in (5). The second point is to take into account the algebraic structure of the Papapetrou field, which in [6, 7] has been shown to be fundamental. The idea is to distinguish between the two possible algebraic types of F_{ab}: (i) The *regular* type, characterized by $\tilde{F}^{ab}\tilde{F}_{ab} \neq 0$. In this case we can pick a Newman-Penrose basis $\{\boldsymbol{k}, \boldsymbol{\ell}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ so that $\tilde{\boldsymbol{F}}$ takes the following *canonical* form

$$\tilde{F}_{ab} = \Upsilon W_{ab} \,, \quad \text{where} \quad \Upsilon = -(\not{\alpha} + i\not{\beta}) \quad \text{and} \quad W_{ab} = -2k_{[a}\ell_{b]} + 2m_{[a}\bar{m}_{b]} \,, \tag{21}$$

and where $\not{\alpha}$ and $\not{\beta}$ are the real eigenvalues of F_{ab}, and $\boldsymbol{k}$ and $\boldsymbol{\ell}$ are its null eigenvectors (principal null directions). (ii) The *singular* type, characterized by $\tilde{F}^{ab}\tilde{F}_{ab} = 0$. Now, we can choose the Newman-Penrose basis so that $\tilde{\boldsymbol{F}}$ can be cast in the form

$$\tilde{F}_{ab} = 2\theta V_{ab} \,, \quad \text{where} \quad V_{ab} = 2k_{[a}m_{b]} \,, \tag{22}$$

where θ is a complex scalar and $\boldsymbol{k}$ gives the only principal null direction.

The NP basis in (21,22) are not completely fixed. The transformations that keep $\tilde{F}_{ab}$ in the canonical form were discussed in [7]. We only mention the fact that whereas in the regular case the eigenvalues $\not{\alpha}$ and $\not{\beta}$ are invariant under these transformation, in the singular case the scalar θ is not, actually we can scale it in an arbitrary way.

We can now study the integrability conditions of the Maxwell equations for the Papapetrou field (20). We are going to consider each algebraic case separately.

3.1. Regular case

Taking into account the following property of W_{ab}

$$W_a{}^b W_{bc} = \mathrm{g}_{ac}\,, \tag{23}$$

one can show that the Maxwell equations (20) are equivalent to the following equations

$$\Upsilon_{,a} = \Upsilon h_a + q_a\,, \tag{24}$$

where

$$h_a = W_{ab;c}W^{bc}\,, \quad q_a = W_{ab}J^b = 2W_a{}^b R_{bc}\xi^c\,. \tag{25}$$

Remarkably, Maxwell equations provide an expression for all the derivatives of the eigenvalues $\not\alpha$ and $\not\beta$ (24) [their expressions in the NP formalism are given in the Appendix A, Eqs. (A.11-A.14)]. In this case the integrability conditions for the Maxwell equations reduce to the integrability conditions for the complex scalar Υ. They will impose conditions on h_a and q_a. Following [19], we divide the problem into two cases: (a) $\boldsymbol{dh} = 0 \Rightarrow h_{[a,b]} = 0$. (b) $h_{[a,b]} \neq 0$. Case (a) represents a particular situation in which a condition on the 2-form W_{ab} has been imposed. The integrability conditions for the complex scalar Υ are simply

$$\boldsymbol{dq} + \boldsymbol{q}\wedge\boldsymbol{h} = 0 \quad\Longrightarrow\quad q_{[a,b]} + q_{[a}h_{b]} = 0\,. \tag{26}$$

This condition is automatically satisfied when $q_a = 0$, or equivalently when $R_{ab}\xi^b = 0$ (the KVF is an eigenvector of the Ricci tensor with zero eigenvalue), which includes the vacuum case. The integrability conditions (26) are first-order equations for $\boldsymbol{q}$ and their integrability conditions, $\boldsymbol{d^2q} = 0$, are identically satisfied by virtue of $\boldsymbol{dh} = 0$. The expressions of this last set of equations in the NP formalism is given in Appendix A, Eqs. (A.34-A.38). Case (b) constitutes the generic situation. The integrability conditions for Υ can be written as follows (see [19] for details)

$$\Omega_{ab}h_{[c,d];e} - h_{[a,b]}[\Omega_{cd;e} - h_c\Omega_{de} + q_c h_{[d,e]}] = 0\,, \tag{27}$$

where

$$\Omega_{ab} = q_{[a,b]} + q_{[a}h_{b]}\,.$$

Another important issue that we must consider is the analysis of the consequences of the relation (7). To that end it is useful to use the complex 2-forms $\{\boldsymbol{U},\boldsymbol{V},\boldsymbol{W}\}$, where the 2-form $\boldsymbol{U}$ is given by

$$U_{ab} = -2l_{[a}\bar{m}_{b]}\,,$$

and $\boldsymbol{W}$ and $\boldsymbol{V}$ have been introduced in (21) and (22) respectively. Using (24), the relation (7) leads to the following three equations

$$U^{ab}\xi^c W_{ab;c} = 0\,, \tag{28}$$

$$V^{ab}\xi^c W_{ab;c} = 0\,, \tag{29}$$

$$\xi^a(\Upsilon h_a + q_a) = 0 \Longrightarrow \xi^a\Upsilon_{,a} = 0\,. \tag{30}$$

These relations mix algebraically the components of the KVF with components of the connection and the Ricci tensor. We have given their explicit form in the NP formalism in Appendix A [Eqs. (A.29-A.31)].

Finally, from (15,16) and (16) one can construct the invariants of the Weyl tensors in terms of the Papapetrou field and the energy-momentum tensor. In the case of vacuum spacetimes and a timelike KVFs, one can also construct the Bel-Robinson *superenergy* (see [20, 21]) associated with the observers following the trajectories of the KVF. These quantities have been used in numerical simulations to track, e.g., the propagation of gravitational waves. In the regular case and vacuum, by virtue of (24), the Bel-Robinson superenergy has the form: $(\not\alpha^2 + \not\beta^2)^2(\text{Connection terms})^2$. For the nonvacuum case, the Bel-Robinson tensor is not longer conserved, but there are generalizations. In the case of spacetimes with vanishing scalar curvature, which include Einstein-Maxwell spacetimes, we have a generalization [22] which also has a positive timelike component (superenergy), where apart from terms of the form showed before we have terms constructed from the energy-momentum tensor and mixed terms of the form: $(\not\alpha^2 + \not\beta^2)(\text{Connection terms})(\text{Components of } T_{ab})$.

3.2. Singular case

The situation in the singular case is completely different. The main reason is that V_{ab} is a singular 2-form which obviously does not satisfy (23). The main consequence of this fact is that the Maxwell equations (20), which now look as follows

$$V_a{}^b\theta_{,b} = -\theta V_a{}^b{}_{;b} + \tfrac{1}{2}J_a\,. \tag{31}$$

does not provide expressions for all the derivatives of the complex scalar θ [see Appendix A.2]. Therefore, we can only talk about the compatibility conditions for the expressions of the known derivatives of θ. Actually, as we have previously said, contrary to what happens with Υ, θ is not an invariant quantity, and we can even scale it arbitrarily. Then, the question of the compatibility conditions is not a key issue.

Finally, the consequences of the relation (7) lead only to two (instead of three as in the regular case) complex equations

$$\xi^c\theta_{,c} + \tfrac{1}{2}\theta U^{ab}\xi^c V_{ab;c} = 0\,, \tag{32}$$

$$W^{ab}\xi^c V_{ab;c} = 0\,. \tag{33}$$

The second one is the same as the second one in the regular case. We have given their explicit form in the NP formalism in Appendix A [Eqs. (A.32,A.33)].

4. General framework

We have seen how the introduction of the Papapetrou field allows us to gain new insights into the study of the consequences of an isometry in spacetime local structure. This was already considered in [6, 7] for the case of vacuum spacetimes. There, the Papapetrou field was used to set up a classification of these spacetimes and to study the possible

relations between the existence and structure of a Killing symmetry and the algebraic classification of spacetimes, the Petrov classification [13, 2]. Moreover, a new formalism was set up, which provides control both on the structure of the Killing symmetry and on the algebraic structure of the spacetime. The way this formalism works was shown in [7] and a number of results were derived (see [14] for the correction of one of the results). In what follows we will set up a framework which takes into account all the information we have presented in the previous sections, and which extends the formalism developed in [7] for vacuum spacetimes to any spacetime, without restrictions on the matter content.

The main idea is to extend the Newman-Penrose (NP) formalism by adding quantities, and their corresponding equations, that describe the Killing symmetry and its associated Papapetrou field. Briefly speaking, the Newman-Penrose formalism [5] is a tetrad formalism where all the tetrad vectors are null vectors. It has reveal itself as a very powerful tool in many different tasks (e.g., exact solutions, study of gravitational radiation, etc.), an in particular, using it we get control on the algebraic structure of the spacetime. The variables and equations in this formalism are (see [2, 23, 24] for definitions and other details): (i) The components of a NP basis $(z_a{}^b) = (k^b, \ell^b, m^b, \bar{m}^b)$ in a coordinate system $\{x^a\}$. The equations for them come from the definition of the associated connection. We obtain them by applying the commutators of the NP basis vectors (equations (7.55)-(7.58) in [2]) to the coordinate system $\{x^a\}$. (ii) The components of the connection, known as spin coefficients. They are 12 complex scalars named as: $(\kappa,\ \sigma,\ \rho,\ \epsilon,\ \nu,\ \lambda,\ \mu,\ \gamma,\ \tau,\ \pi,\ \alpha,\ \beta)$. The equations they obey come from the integrability conditions for the variables in (i), the so-called NP equations (equations (7.28)-(7.45) in [2]), which are just the expressions of the components, in the NP basis, of the Riemann tensor in terms of the complex connection. (iii) Riemann curvature variables. These are divided into Weyl tensor components, described in the NP formalism by five complex scalars Ψ_A $(A = 0, \ldots, 4)$; components of the traceless Ricci tensor, given by complex scalars Φ_{XY} $(X, Y = 0, 1, 2)$ satisfying $\Phi_{XY} = \bar{\Phi}_{YX}$; and the scalar curvature, Λ. The equations for these quantities are the second Bianchi identities (equations (7.61)-(7.71) in [2, 25]), which are at the same time integrability conditions for the NP equations. With these equations we get a closed system of equations for the whole set of variables. Finally, when we get a specific description of the energy-momentum content through a set of matter fields, we can express the quantities Φ_{XY} and Λ in terms of these fields and to add the equations for the matter fields. A good example is the case of electromagnetic fields (see, e.g., [2]).

Our formalism is an extension of the NP formalism based on the following two ideas: (A) To adapt the NP formalism as much as possible to the existence of a Killing symmetry. (B) To extend the NP formalism by adding new variables and equations characterizing the Killing symmetry.

The best way of implementing the point (A) is through the choice of the NP basis $\{\boldsymbol{k}, \boldsymbol{\ell}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$. We will choose the NP basis to be adapted to the algebraic structure of the Killing symmetry. That is, in the case regular case we will choose a NP basis in

which the Papapetrou field $\boldsymbol{F}$ takes the canonical form (21), and in the singular case we choose a NP basis in which $\boldsymbol{F}$ has the canonical form (22). Then, in our framework all the quantities and equations of the NP formalism will be written in such an adapted NP basis. With regard to point (B), we extend the NP formalism by adding, to the sets described above [(i)-(iii)], the following variables and equations that describe the Killing symmetry: (iv) The components of the KVF in the chosen adapted NP basis. They are introduced through the following expressions

$$\xi_k = k^a\xi_a\,, \quad \xi_l = \ell^a\xi_a\,, \quad \xi_m = m^a\xi_a\,, \quad \xi_{\bar{m}} = \bar{m}^a\xi_a\,, \tag{34}$$

where ξ_k and ξ_l are real scalars, and ξ_m and $\xi_{\bar{m}}$ are complex and related by $\xi_{\bar{m}} = \bar{\xi}_m$. Then,

$$\boldsymbol{\xi} = -\xi_l\boldsymbol{k} - \xi_k\boldsymbol{\ell} + \bar{\xi}_m\boldsymbol{m} + \xi_m\bar{\boldsymbol{m}}\,. \tag{35}$$

The equations for the variables (ξ_k, ξ_l, ξ_m) are obtained from the definition of the Papapetrou field (8). Their form once projected onto an adapted NP basis is shown in equations (A.1-A.10) of Appendix A.1. (v) The components of the Papapetrou field in the adapted NP basis. In the regular case these components are completely described by the complex quantity $\Upsilon = -(\not{\alpha} + i\not{\beta})$, and in the singular case by the complex scalar θ. The equations for these quantities come from the projection of Maxwell equations (4) onto the adapted NP basis. We have given them in Appendix A.2. The equations for the regular case are (A.11-A.14), and the equations for the singular case are (A.15-A.18).

Another set of equations that we have to take into account to complete the formalism are those coming from Eqs. (6,7). As we have seen before, in the regular case they lead to 3 complex algebraic equations [Eqs. (28-30)]. When we express the equations in the NP formalism, using a basis adapted to the Papapetrou field, they become equations involving the components of the Killing, the spin coefficients and components of the Ricci tensor [see Eqs. (A.29-A.31) in Appendix A]. In the singular case we only get two complex equations [Eqs. (32,33)], and one of them, Eq. (32), coincides with one of the regular case, namely Eq. (29). Their expressions in the NP formalism are similar to those in the regular case, with the only difference that now they involve derivatives of θ [see Eqs. (A.32,A.33) in Appendix A].

So far we have presented the formalism, now let us see what is the best way of using it. Following the discussion in section 2, the integrability conditions for the components of the KVF are a good starting point since they lead to an expression for the Weyl tensor [see Eqs. (17,18)]. That means that we can express the complex scalars Ψ_A $(A = 0, \ldots, 4)$ in terms of the rest of quantities of the formalism. In Appendix A.3 we give these expressions for the regular [Eqs. (A.19-A.23)] and singular [Eqs. (A.24-A.28)] cases.

Remarkably, the expressions for the regular case are algebraic in the spin coefficients, the eigenvalues of the Papapetrou field $\not{\alpha} + i\not{\beta}$, the KVF components, and the components of the Ricci tensor. The reason for that comes from the Maxwell equations, which are equivalent to the equations (24). Then, the quantity $F_{ab;c}$ can be written in terms of F_{ab}, the connection and the Ricci tensor, hence the algebraic

expressions for Ψ_A. In the singular case some derivatives of the complex scalar θ appear in the Weyl tensor but, as we have mentioned before, we can arbitrarily scale this quantity by using the remaining freedom in the choice of the NP basis.

The main consequence of having an expression for the Weyl tensor is that we do not need the equations for it. Then, from the second Bianchi identities we only need to consider the subset corresponding to the energy-momentum conservation equations

$$G^{ab}{}_{;b} = 0 \quad \Rightarrow \quad T^{ab}{}_{;b} = 0\,. \tag{36}$$

Then, the situation after introducing the information coming from the integrability conditions for the components of the KVF is a simplified framework, where we can forget about the components of the Weyl tensor and their equations.

Apart from the equations we have listed above, we must take into account the integrability/compatibility conditions for the components of the Papapetrou field F_{ab}, and the consequences of the fact that F_{ab} is invariant under the 1-parameter family of diffeomorphisms generated by the KVF $\boldsymbol{\xi}$ [Eqs. (6,7)]. The first group of equations have been considered in section 3. In the particular case when the Papapetrou field is regular and $\boldsymbol{dh} = 0$, we have seen that they lead to five independent equations only for the connection, the spin coefficients, which complement the Newman-Penrose equations. We have given them in the Appendix A [Eqs. (A.34-A.38)]. In the general regular case the integrability conditions are (27). When the Papapetrou field is singular, we can only talk about compatibility conditions.

Finally, in the case that we know the particular description for the energy-momentum content, that is, the matter fields and the field equations they satisfy, we can adopt the following point of view. First, we can add the matter fields as new variables of the formalism, and the field equations they obey as equations of the formalism. Then, we will get an expression, via the energy-momentum tensor, for the components of the Ricci tensor, $\Phi_{XY}\,, \Lambda$, in terms of the matter fields and, probably, of their derivatives. The advantage of this is that now we can ignore the quantities Φ_{XY} and the equations that they satisfy, namely the contracted second Bianchi identities (36). The last point would be to consider the integrability/compatibility conditions for the equations of the matter fields, in a similar way as we have done with the equations for the Papapetrou field in the previous section. In the next section, we illustrate this alternative procedure with the case of spacetimes with a gravitating electromagnetic field.

5. Particular case: gravitating electromagnetic field

To illustrate the framework we have described above we are going to consider the particular case of spacetimes whose energy-momentum content corresponds to an electromagnetic field, i.e., we have a gravitating electromagnetic field. We will call it H_{ab}, to distinguish it from the Papapetrou field F_{ab}. The self-dual field $\tilde{H}_{ab}$ will satisfy Maxwell equations

$$\tilde{H}^{ab}{}_{;b} = S^a\,, \tag{37}$$

where S^a are the electromagnetic sources. In general, H_{ab} will not be aligned with the Papapetrou field, so in the NP basis adapted to the Papapetrou field it will take the following generic form

$$\tilde{H}_{ab} = \Phi_0 U_{ab} + \Phi_1 W_{ab} + \Phi_2 V_{ab}\,, \tag{38}$$

where Φ_X $(X = 0, 1, 2)$ are the components of $\tilde{H}_{ab}$ in such a basis. The energy-momentum tensor of the spacetime will be given by

$$T_{ab} = \tfrac{1}{2}\tilde{H}_a{}^c \bar{\tilde{H}}_{cb} \quad \Rightarrow \quad \mathrm{g}^{ab}T_{ab} = 0\,. \tag{39}$$

Therefore, the components of the Ricci tensor in terms of the components of H_{ab} are (see, e.g., [2])

$$\Phi_{XY} = \Phi_X \bar{\Phi}_Y\,, \qquad \Lambda = 0\,. \tag{40}$$

The NP equations for Φ_X can be found, e.g., in [2] [Eqs. (7.46)-(7.49)]. Following the framework described above, we have to include the quantities Φ_X and their equations, but then we can forget about Φ_{XY}, which are given by (40), and their equations, the contracted second Bianchi identities (36). So we have to concentrate on the Maxwell equations for the gravitating electromagnetic field. If we insert (38) into Maxwell equations (37) we obtain

$$\begin{aligned} &\Phi_{1,a} = \Phi_1 h_a + \hat{q}_a\,, \\ &\hat{q}_a = W_{ab}S^b + W_{ac}(\Phi_0 U^{bc} + \Phi_2 V^{bc})_{;b}\,, \end{aligned} \tag{41}$$

where h_a is given in (25). Again we can follow [19] to study the integrability conditions for (41), and the classification used for the case of the Papapetrou field depending on whether $h_{[a,b]}$ vanishes or not. Here, it is important to remark that this 1-form h_a is the same for both the Papapetrou field F_{ab} and the gravitating electromagnetic field H_{ab}. This means we can analyze simultaneously the integrability conditions for F_{ab} (in the regular case) and H_{ab}. Then, when $\boldsymbol{dh} = 0$ the integrability conditions are given by (26) and

$$\hat{q}_{[a,b]} + \hat{q}_{[a}h_{b]} = 0\,.$$

The condition $\boldsymbol{dh} = 0$ takes place in the case of Einstein-Maxwell spacetimes ($S^a = 0$) where F_{ab} and H_{ab} are regular and aligned (see the particular case below in the subsection 5.1).

In the generic case, when $h_{[a,b]} \neq 0$, the integrability conditions are given by (27) and

$$\hat{\Omega}_{ab}h_{[c,d];e} - h_{[a,b]}[\hat{\Omega}_{cd;e} - h_c\hat{\Omega}_{de} + \hat{q}_c h_{[d,e]}] = 0\,,$$

where

$$\hat{\Omega}_{ab} = \hat{q}_{[a,b]} + \hat{q}_{[a}h_{b]}\,.$$

5.1. A particular case: Einstein-Maxwell spacetimes with alignment between the Papapetrou and the electromagnetic field

Let us now consider a very particular case, in which the gravitating electromagnetic field is source-free ($S^a = 0$) and it is aligned with the Papapetrou field in the following sense

$$\tilde{H}_{ab} = \zeta\,\tilde{F}_{ab}\,, \tag{42}$$

where ζ is an arbitrary complex scalar. In the case of a regular Papapetrou field, H_{ab} will have two principal directions which coincide with those of F_{ab}. Moreover, we will have $\Phi_0 = \Phi_2 = 0$ in (38), which means that now we have $\hat{q}_a = 0$. Therefore, from (40) it follows that

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{12} = \Phi_{22} = 0 = \Lambda\,.$$

Or in other words, the only non-zero component of the Ricci tensor is $\Phi_{11} = \Phi_1\bar{\Phi}_1$. Comparing the equations for the Papapetrou and the electromagnetic fields we get an equation for the proportionality factor ζ

$$\boldsymbol{d}\mathbf{log}\,\boldsymbol{\zeta} = -\Upsilon^{-1}\boldsymbol{q}\,.$$

The integrability conditions for ζ are the same as those for Υ, i.e. Eqs. (26), corresponding to the case $\boldsymbol{dh} = 0$. The interesting characteristic of this particular case is that if we look at the expressions for the Weyl tensor components, Eqs. (A.19-A.23), we see that they are formally the same as those for the vacuum case [7]. Also the NP equations coming from (6), Eqs.(A.29-A.31), have the same form as in the vacuum case. Finally, the particular case where the proportional factor is constant, $\boldsymbol{d\zeta} = 0$, is empty since it implies either $\Upsilon = 0$ or $\zeta = 0$.

An interesting example in which we find this kind of alignment is the case of the Kerr-Newman metric [26, 2], which in Boyer-Linquist coordinates reads

$$ds^2 = -\frac{\Delta}{\varrho^2}(dt - a\sin^2\theta d\varphi)^2 + \frac{\sin^2\theta}{\varrho^2}[(r^2+a^2)d\varphi - a dt]^2 + \frac{\varrho^2}{\Delta}dr^2 + \varrho^2 d\theta^2\,,$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2$ and $\varrho^2 = r^2 + a^2\cos^2\theta$, being M, a and Q constants denoting respectively the mass, angular momentum, and charge of the Kerr-Newman black hole. It is easy to see that the Papapetrou field associated with the timelike KVF $\boldsymbol{\xi_t} = \boldsymbol{\partial/\partial t}$ has the following form

$$\begin{aligned}\boldsymbol{F} = {}& \frac{2}{\varrho^4}[M(r^2 - a^2\cos^2\theta) - rQ^2](\boldsymbol{dt} - a\sin^2\theta\boldsymbol{d\varphi})\wedge\boldsymbol{dr} \\ &+ \frac{a(2Mr - Q^2)\sin(2\theta)}{\varrho^4}\boldsymbol{d\theta}\wedge(a\boldsymbol{dt} - (r^2+a^2)\boldsymbol{d\varphi})\,.\end{aligned} \tag{43}$$

The Kerr-Newman metric is a solution of the Einstein-Maxwell system of field equations, and the electromagnetic field H_{ab} generating the energy-momentum distribution is given by

$$\begin{aligned}\boldsymbol{H} = {}& -\sqrt{2}\,\frac{Q}{\varrho^4}(r^2 - a^2\cos^2\theta)(\boldsymbol{dt} - a\sin^2\theta\boldsymbol{d\varphi})\wedge\boldsymbol{dr} \\ & -\sqrt{2}\,\frac{rQ}{\varrho^4}a\sin(2\theta)\boldsymbol{d\theta}\wedge(a\boldsymbol{dt} - (r^2+a^2)\boldsymbol{d\varphi})\,,\end{aligned} \tag{44}$$

which can be generated by the following vector potential

$$\boldsymbol{A} = -\sqrt{2}\,\frac{rQ}{\varrho^2}(\boldsymbol{dt} - a\sin^2\theta\boldsymbol{d\varphi}) = -\frac{\sqrt{2}\,rQ}{2Mr - Q^2}(\boldsymbol{\xi_t} + \boldsymbol{dt})\,.$$

From the expressions for $\boldsymbol{F}$ [Eq. (43)] and $\boldsymbol{H}$ [Eq. (44)] we can construct the respective self-dual 2-forms, and from (42) we can find ζ. The result is

$$\zeta = -\frac{Q}{\sqrt{2}}\frac{r + ia\cos\theta}{M(r + ia\cos\theta) - Q^2}\,.$$

Since $\boldsymbol{F}$ and $\boldsymbol{H}$ are aligned in the sense of (42) they have the same principal null directions. These directions, which also coincide with the two multiple directions of the spacetime (of the Weyl tensor), are given by the following vector fields

$$\boldsymbol{K} = -\frac{\Delta}{\varrho^2}\boldsymbol{dt} + \boldsymbol{dr} + \frac{a\sin^2\theta\Delta}{\varrho^2}\boldsymbol{d\varphi}\,, \qquad \boldsymbol{L} = -\frac{\Delta}{\varrho^2}\boldsymbol{dt} - \boldsymbol{dr} + \frac{a\sin^2\theta\Delta}{\varrho^2}\boldsymbol{d\varphi}\,.$$

Note that these vector fields are not normalized, i.e. $K^aL_a \neq -1$. From them we can compute the complex scalar Υ containing the eigenvalues $\not{\alpha}$ and $\not{\beta}$ [see Eq. (21)]:

$$\Upsilon = -2\frac{(r + ia\cos\theta)}{\varrho^4}\left[M(r + ia\cos\theta) - Q^2\right]\,.$$

Using all this information we can finally get $\boldsymbol{q}$ and $\boldsymbol{h}$:

$$\boldsymbol{q} = -2\frac{Q^2}{\varrho^4}\boldsymbol{d(r + ia\cos\theta)}\,, \qquad \boldsymbol{h} = -2\boldsymbol{d}\ln\boldsymbol{(r - ia\cos\theta)}\,.$$

As expected, we have $\boldsymbol{dh} = 0$.

In the case of a singular Papapetrou field, H_{ab} will be also singular and with the same principal direction. In this case we have $\Phi_0 = \Phi_1 = 0$ in (38), hence we find that

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{11} = \Phi_{12} = 0 = \Lambda\,.$$

Then, the only non-zero component of the Ricci tensor is $\Phi_{22} = \Phi_2\bar{\Phi}_2$. From the expressions for the Ψ_A $(A = 0, ..., 4)$ given in the Appendix A we have

$$\Psi_0 = \Psi_1 = \Psi_2 = 0\,,$$

which means that the spacetime is not only algebraically special, as the Mariot-Robinson theorem [27, 28] tells us, but also that it must be either of the Petrov type III or N. In other words, the only principal null direction, $\boldsymbol{k}$, must have at least multiplicity three. This generalizes the result found in [7] for vacuum spacetimes. Here, when the proportional factor ζ is constant, one can see that this leads to the additional condition $\rho = 0$, which means that we are within the Kundt class of spacetimes [29].

6. Remarks and discussion

In this paper we have discussed the consequences of a symmetry on the local structure of the spacetime. We have adopted the point of view introduced in [6, 7], where the Papapetrou field plays a central role. We have also generalized the formalism introduced in [7] for vacuum spacetimes to the general case. In our discussion we have shown that

the advantages of this type of formalism lie on the consideration of the following two essential points: first, that the spacetimes of General Relativity are four-dimensional, which we have shown leads to an expression for the components of the Weyl tensor through the integrability conditions for the components of the KVF. In this line, the formalism we have presented is based on the Newman-Penrose formalism, in which the fact that the spacetime is four-dimensional is present. The second important point is that we have written the NP equations in a basis adapted to the algebraic structure of the Papapetrou field, and we have added variables, with their corresponding equations, that describe the KVF as well as its associated Papapetrou field. By one hand this simplifies the equations, specially if we want to study cases in which there are alignments between the principal directions of the Weyl tensor and the Papapetrou field (see [7, 14]). On the other hand, these new variables provide a new point of view to study the integrability conditions. Finally, using the Papapetrou field we introduce information about the particular characteristics of the symmetry, for instance, when we decide whether a KVF has a regular or a singular Papapetrou field.

We have illustrated the formalism in the case of spacetimes with a gravitating electromagnetic field, where we have described how the formalism should be applied and we have produced some new results. The fact that this formalism can be applied to any spacetime with a Killing symmetry means that we can use it with a number of interesting physical situations, and in this sense it could be used to complement or in combination with other methods of study, like perturbative schemes or numerical computations of spacetimes.

Finally, it is important to remark that here we have dealt only with the case of Killing symmetries however, the structure of the formalism and the way in which it has been set up, can be easily translated to other kinds of symmetries. Actually, there are some other works in which this has already been done. The case of homotheties in vacuum spacetimes has been studied in [15], and in [16] it was applied to conformal Killing symmetries.

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Appendix A. Formalism for spacetimes with a Killing symmetry. NP equations

In this Appendix we give the equations of the general framework present above using the Newman-Penrose formalism.

Appendix A.1. Equations for the components of the KVF

The equations for the components of the KVF [see Eqs. (34,35)] are obtained by projecting Eq. (8) onto an adapted NP basis. The result is the following (they are the same as those found in [7] for the vacuum case):

$$
\begin{aligned}
&D\xi_k - (\epsilon+\bar{\epsilon})\xi_k + \bar{\kappa}\xi_m + \kappa\bar{\xi}_m = 0\,, && \text{(A.1)}\\
&\triangle\xi_k - (\gamma+\bar{\gamma})\xi_k + \bar{\tau}\xi_m + \tau\bar{\xi}_m = \tfrac{1}{2}\not{\alpha}\,, && \text{(A.2)}\\
&\delta\xi_k - (\bar{\alpha}+\beta)\xi_k + \bar{\rho}\xi_m + \sigma\bar{\xi}_m = 0\,, && \text{(A.3)}\\
&D\xi_l + (\epsilon+\bar{\epsilon})\xi_l - \pi\xi_m - \bar{\pi}\bar{\xi}_m = -\tfrac{1}{2}\not{\alpha}\,, && \text{(A.4)}\\
&\triangle\xi_l + (\gamma+\bar{\gamma})\xi_l - \nu\xi_m - \bar{\nu}\bar{\xi}_m = 0\,, && \text{(A.5)}\\
&\delta\xi_l + (\bar{\alpha}+\beta)\xi_l - \mu\xi_m - \bar{\lambda}\bar{\xi}_m = \tfrac{1}{2}\bar{\theta}\,, && \text{(A.6)}\\
&D\xi_m - (\epsilon-\bar{\epsilon})\xi_m - \bar{\pi}\xi_k + \kappa\xi_l = 0\,, && \text{(A.7)}\\
&\triangle\xi_m - (\gamma-\bar{\gamma})\xi_m - \bar{\nu}\xi_k + \tau\xi_l = -\tfrac{1}{2}\bar{\theta}\,, && \text{(A.8)}\\
&\delta\xi_m + (\bar{\alpha}-\beta)\xi_m - \bar{\lambda}\xi_k + \sigma\xi_l = 0\,, && \text{(A.9)}\\
&\bar{\delta}\xi_m - (\alpha-\bar{\beta})\xi_m - \bar{\mu}\xi_k + \rho\xi_l = -\tfrac{1}{2}i\not{\beta}\,, && \text{(A.10)}
\end{aligned}
$$

where $(D, \triangle, \delta, \bar{\delta})$ are the directional derivatives along the NP basis vectors

$$D = k^a\partial_a\,, \quad \triangle = \ell^a\partial_a\,, \quad \delta = m^a\partial_a\,, \quad \bar{\delta} = \bar{m}^a\partial_a\,.$$

These equations include at the same time both the regular and singular cases. The regular case is obtained by taking $\theta = 0$ and for the singular case we need to take $\not{\alpha} = \not{\beta} = 0$.

Appendix A.2. Equations for the components of the Papapetrou field

They are the projection of the Maxwell equations (4) onto the adapted NP basis. In the regular case they are equations for the complex quantities $\not{\alpha} + i\not{\beta}$, actually equations (24) :

$$
\begin{aligned}
&D(\not{\alpha}+i\not{\beta}) = 2\rho(\not{\alpha}+i\not{\beta}) - 4\left\{\xi_k(\Phi_{11}-3\Lambda) + \xi_l\Phi_{00} - \xi_m\bar{\Phi}_{01} - \bar{\xi}_m\Phi_{01}\right\}\,, && \text{(A.11)}\\
&\triangle(\not{\alpha}+i\not{\beta}) = -2\mu(\not{\alpha}+i\not{\beta}) + 4\left\{\xi_k\Phi_{22} + \xi_l(\Phi_{11}-3\Lambda) - \xi_m\bar{\Phi}_{12} - \bar{\xi}_m\Phi_{12}\right\}\,, && \text{(A.12)}\\
&\delta(\not{\alpha}+i\not{\beta}) = 2\tau(\not{\alpha}+i\not{\beta}) - 4\left\{\xi_k\Phi_{12} + \xi_l\Phi_{01} - \xi_m(\Phi_{11}+3\Lambda) - \bar{\xi}_m\Phi_{02}\right\}\,, && \text{(A.13)}\\
&\bar{\delta}(\not{\alpha}+i\not{\beta}) = -2\pi(\not{\alpha}+i\not{\beta}) + 4\left\{\xi_k\bar{\Phi}_{12} + \xi_l\bar{\Phi}_{01} - \xi_m\bar{\Phi}_{02} - \bar{\xi}_m(\Phi_{11}+3\Lambda)\right\}\,. && \text{(A.14)}
\end{aligned}
$$

Then, we have an expression for all the directional derivatives of $\not{\alpha}$ and $\not{\beta}$. In the singular case, Maxwell equations reduce to equations (31) for θ, which once project onto the adapted NP basis are

$$
\begin{aligned}
&D\theta = (\rho-2\epsilon)\theta + 2\left\{\xi_k\bar{\Phi}_{12} + \xi_l\bar{\Phi}_{01} - \xi_m\bar{\Phi}_{02} - \bar{\xi}_m(\Phi_{11}+3\Lambda)\right\}\,, && \text{(A.15)}\\
&\delta\theta = (\tau-2\beta)\theta + 2\left\{\xi_k\Phi_{22} + \xi_l(\Phi_{11}-3\Lambda) - \xi_m\bar{\Phi}_{12} - \bar{\xi}_m\Phi_{12}\right\}\,, && \text{(A.16)}\\
&\tfrac{1}{2}\kappa\theta = \xi_k(\Phi_{11}-3\Lambda) + \xi_l\Phi_{00} - \xi_m\bar{\Phi}_{01} - \bar{\xi}_m\Phi_{01}\,, && \text{(A.17)}\\
&\tfrac{1}{2}\sigma\theta = \xi_k\Phi_{12} + \xi_l\Phi_{01} - \xi_m(\Phi_{11}+3\Lambda) - \bar{\xi}_m\Phi_{02}\,. && \text{(A.18)}
\end{aligned}
$$

In this case, we only get two directional derivatives of θ, in the directions of $\boldsymbol{k}$ and $\boldsymbol{m}$. The other two equations are algebraic relations, from where we can recover the Goldberg-Sachs and Mariot-Robinson theorems [30, 27, 28] in the particular case of spacetimes with a KVF.

Appendix A.3. Integrability conditions for the components of the KVF

The integrability conditions for the components of $\boldsymbol{\xi}$ lead to an expression for the Weyl tensor [Eqs. (17, 18)]. From it we can compute the Weyl complex scalars Ψ_A. In the regular case, we get the following expressions:

$$\Psi_0 = \frac{\not\alpha + i\not\beta}{N}(\kappa\xi_m - \sigma\xi_k) - \frac{2}{N}(\xi_m^2\Phi_{00} - 2\xi_k\xi_m\Phi_{01} + \xi_k^2\Phi_{02})\,, \tag{A.19}$$

$$\Psi_1 = \frac{\not\alpha + i\not\beta}{N}(\kappa\xi_l - \sigma\bar{\xi}_m) - \frac{2}{N}\left\{\xi_l\xi_m\Phi_{00} - (\xi_k\xi_l + \xi_m\bar{\xi}_m)\Phi_{01} + \xi_k\bar{\xi}_m\Phi_{02}\right\}\,, \tag{A.20}$$

$$\Psi_2 = \frac{\not\alpha + i\not\beta}{N}(\rho\xi_l - \tau\bar{\xi}_m) - \frac{2}{N}\left\{\xi_l^2\Phi_{00} - 2\xi_l\bar{\xi}_m\Phi_{01} + \bar{\xi}_m^2\Phi_{02}\right\} - 2\Lambda\,, \tag{A.21}$$

$$\Psi_3 = \frac{\not\alpha + i\not\beta}{N}(\nu\xi_k - \lambda\xi_m) - \frac{2}{N}\left\{\xi_l\xi_m\bar{\Phi}_{02} - (\xi_k\xi_l + \xi_m\bar{\xi}_m)\bar{\Phi}_{12} + \xi_k\bar{\xi}_m\Phi_{22}\right\}\,, \tag{A.22}$$

$$\Psi_4 = \frac{\not\alpha + i\not\beta}{N}(\nu\bar{\xi}_m - \lambda\xi_l) - \frac{2}{N}(\xi_l^2\bar{\Phi}_{02} - 2\xi_l\bar{\xi}_m\bar{\Phi}_{12} + \bar{\xi}_m^2\Phi_{22})\,. \tag{A.23}$$

Here we have used that $N = -2\xi_k\xi_l + 2\xi_m\bar{\xi}_m$. In the singular case, the complex scalars Ψ_A can be written as follows:

$$\Psi_0 = -\frac{2}{N}(\xi_m^2\Phi_{00} - 2\xi_k\xi_m\Phi_{01} + \xi_k^2\Phi_{02})\,, \tag{A.24}$$

$$\Psi_1 = -\frac{2}{N}\left\{\xi_l\xi_m\Phi_{00} - (\xi_k\xi_l + \xi_m\bar{\xi}_m)\Phi_{01} + \xi_k\bar{\xi}_m\Phi_{02}\right\}\,, \tag{A.25}$$

$$\Psi_2 = -\frac{2}{N}\left\{\xi_l^2\Phi_{00} - 2\xi_l\bar{\xi}_m\Phi_{01} + \bar{\xi}_m^2\Phi_{02}\right\} - 2\Lambda\,, \tag{A.26}$$

$$\Psi_3 = -\frac{1}{N}(\rho\xi_l - \tau\bar{\xi}_m)\theta - \frac{2}{N}(\xi_l^2\bar{\Phi}_{01} - 2\xi_l\bar{\xi}_m\Phi_{11} + \bar{\xi}_m^2\Phi_{12})\,, \tag{A.27}$$

$$\Psi_4 = \frac{1}{N}\left\{\bar{\xi}_m(\triangle + 2\gamma) - \xi_l(\bar{\delta} + 2\alpha)\right\}\theta - \frac{2}{N}(\xi_l^2\bar{\Phi}_{02} - 2\xi_l\bar{\xi}_m\bar{\Phi}_{12} + \bar{\xi}_m^2\Phi_{22})\,. \tag{A.28}$$

Appendix A.4. Invariance of F_{ab}

From the Ricci identities we get the equations for the Papapetrou field, the components of the Weyl tensor and some extra relations, equations (28-30) in the regular case and equations (32,33) in the singular case. These relations come from the invariance of F_{ab} [Eq. (6)]. In the NP formalism, the equations for the regular case are:

$$\nu\xi_k + \pi\xi_l - \lambda\xi_m - \mu\bar{\xi}_m = 0\,, \tag{A.29}$$

$$\tau\xi_k - \rho\xi_m + \kappa\xi_l - \sigma\bar{\xi}_m = 0\,, \tag{A.30}$$

$$\mu\xi_k - \rho\xi_l - \pi\xi_m + \tau\bar{\xi}_m = \frac{2}{\not\alpha + i\not\beta}(\xi_k^2\Phi_{22} - \xi_l^2\Phi_{00} - 2\xi_k\xi_m\bar{\Phi}_{12} + 2\xi_l\bar{\xi}_m\Phi_{01} + \xi_m^2\Phi_{20} - \bar{\xi}_m^2\Phi_{02})\,. \tag{A.31}$$

And for the singular case:

$$\tau\xi_k - \rho\xi_m + \kappa\xi_l - \sigma\bar{\xi}_m = 0\,, \tag{A.32}$$

$$\begin{aligned}\xi_k(\triangle + 2\gamma)\theta - \xi_m(\bar{\delta} + 2\alpha)\theta + (\rho\xi_l - \tau\bar{\xi}_m)\theta = \\ -2\xi_l^2\bar{\Phi}_{01} + 2\xi_l\xi_m\bar{\Phi}_{02} - 2(\xi_k\xi_l + \xi_m\bar{\xi}_m)\bar{\Phi}_{12} + 4\xi_l\bar{\xi}_m\Phi_{11} - 2\bar{\xi}_m^2\Phi_{12} + 2\xi_k\bar{\xi}_m\Phi_{22}\,.\end{aligned} \tag{A.33}$$

Appendix A.5. NP form of the equations $\boldsymbol{dh} = 0$

Within the regular case, a particular situation of interest corresponds to the subcase defined by the condition $\boldsymbol{dh} = 0$. These equations can written in the NP formalism in the following form:

$$(D + \epsilon + \bar{\epsilon})\mu + (\triangle - \gamma - \bar{\gamma})\rho - \pi\bar{\pi} + \tau\bar{\tau} = 0\,, \tag{A.34}$$

$$(\delta + \bar{\pi} - \bar{\alpha} - \beta)\rho - (D - \bar{\rho} - \epsilon + \bar{\epsilon})\tau + \mu\kappa - \pi\sigma = 0\,, \tag{A.35}$$

$$(\bar{\delta} - \alpha - \bar{\beta})\rho + (D + \epsilon - \bar{\epsilon})\pi + \mu\bar{\kappa} + \tau\bar{\sigma} = 0\,, \tag{A.36}$$

$$(\delta + \bar{\alpha} + \beta)\mu + (\triangle - \gamma + \bar{\gamma})\tau - \rho\bar{\nu} - \pi\bar{\lambda} = 0\,, \tag{A.37}$$

$$(\triangle + \bar{\mu} + \gamma - \bar{\gamma})\pi - (\bar{\delta} - \bar{\tau} + \alpha + \bar{\beta})\mu - \tau\lambda + \rho\nu = 0\,. \tag{A.38}$$

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[25] Note the misprint in equation (7.63) of [2]: instead of "$\ldots - \triangle\Psi_4 \ldots$" it should read "$\ldots - D\Psi_4 \ldots$."

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